Transport-Corrected Diffusion Theory for Image Reconstruction in Optical Tomography

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ABSTRACT

We introduce a set of corrections to the integral equations of scattering theory within the diffusion approximation to the radiative transport equation. We use this result to obtain an imaging reconstruction algorithm for optical tomography with spatial resolution below the transport mean free path.

Keywords: Optical tomography, radiative transport, inverse scattering

There has been considerable recent interest in the development of optical methods for tomographic imaging.1 The physical problem that is considered is to recover the optical properties of the interior of an inhomogeneous medium from measurements taken on its surface. The starting point for the mathematical formulation of this inverse scattering problem (ISP) is a model for the propagation of light, typically taken to be the diffusion approximation (DA) to the radiative transport equation (RTE). The DA is valid when the energy density of the optical field varies slowly on the scale of the transport mean free path $\ell^*$. The DA breaks down in optically thin layers, near boundary surfaces, or near the source. One or more of these conditions are encountered in biomedical applications such as imaging of small animals2 or of functional activity in the brain.3

Within the accuracy of the DA, reconstruction algorithms based on both numerical4 and analytic solutions5-7 to the ISP have been described. Regardless of the method of inversion, the spatial resolution of reconstructed images is expected to be limited to $\ell^*$. This is due to the intertwined effects of the ill-posedness of the ISP6 and intrinsic inaccuracies of the DA.8 Improvements in resolution beyond the limit of $\ell^*$ may be achieved, in principle, by making use of inversion methods based on the RTE.3,7 The ISP for the RTE is mathematically complex and has been the subject of several recent studies.9-11 In this paper, we introduce an alternative to the full RTE, namely a set of corrections to the standard integral equations of scattering theory within the DA. Using this result, it is possible to reconstruct superresolved images whose spatial resolution is less than $\ell^*$.

We begin by considering the propagation of multiply-scattered light in an inhomogenous medium characterized by an absorption coefficient $\mu_a(r)$. In what follows, we will neglect the contribution of ballistic photons and consider only diffuse photons whose specific intensity $I(r, \hat{s})$ at the point $r$ in the direction $\hat{s}$ is taken to obey the time-independent RTE

$$\hat{s} \cdot \nabla I(r, \hat{s}) + (\mu_a + \mu_g)I(r, \hat{s}) - \mu_s \int d^2s' A(\hat{s}, \hat{s}') I(r, \hat{s}') = S(r, \hat{s}) ,$$

where $\mu_a$ is the scattering coefficient, $A(\hat{s}, \hat{s}')$ is the scattering kernel, and $S(r, \hat{s})$ is the source. The change in specific intensity due to spatial fluctuations in $\mu_a(r)$ can be obtained from the integral equation

$$\phi(r_1, \hat{s}_1; r_2, \hat{s}_2) = \int d^3rd^2s G(r_1, \hat{s}_1; r, \hat{s}) G(r, \hat{s}; r_2, \hat{s}_2) \delta \mu_a(r) .$$
Here the data function $\phi(r_1, s_1; r_2, s_2)$ is proportional, to lowest order in $\delta\mu_\alpha$, to the change in specific intensity relative to a reference medium with absorption $\mu_\alpha^0$, $G$ is the Green's function for (1) with $\mu_\alpha = \mu_\alpha^0$, $\delta\mu_\alpha(r) = \mu_\alpha(r) - \mu_\alpha^0$, $r_1, s_1$ denote the position and direction of a unidirectional point source, and $r_2, s_2$ denote the position and direction of a unidirectional point detector.

We now show that the integral equation (2) may be used to obtain corrections to the usual formulation of scattering theory within the DA. To proceed, we note that, following Ref. 2, the Green's function $G(r, s; r', s')$ may be expanded in angular harmonics of $\hat{s}$ and $\hat{s}'$:

$$G(r, \hat{s}; r', \hat{s}') = \frac{c}{4\pi} \frac{1 + \ell^* \hat{s} \cdot \nabla_r}{1 - \ell^* \hat{s'} \cdot \nabla_{r'}} G(r, r') ,$$  

(3)

where $\ell^* = 1/[\mu_\alpha^0 + (1-g)\mu_\alpha]$ with $g$ being the anisotropy of the scattering kernel $A$. The Green's function $G(r, r')$ satisfies the diffusion equation $(-D\nabla^2 + \alpha_0) G(r, r') = \delta(r - r')$, where the diffusion coefficient $D = 1/3c\ell^*$ and $\alpha_0 = c\ell^* \mu_\alpha^0$. In addition, the Green's function must satisfy boundary conditions on the surface of the medium (or at infinity in the case of free boundaries). In general we will consider boundary conditions of the form $G(r, r') + \ell \hat{n} \cdot \nabla G(r, r') = 0$, where $\hat{n}$ is the outward unit normal to the surface bounding the medium and $\ell$ is the extrapolation distance. Making use of (3) and performing the angular integration over $\hat{s}$ in (2) we obtain

$$\phi(r_1, s_1; r_2, s_2) = c \frac{\Delta_1 \Delta_2}{4\pi} \int d^3 r \left( G(r_1, r) G(r, r_2) - \frac{\ell^*}{3} \nabla_r G(r_1, r) \cdot \nabla_r G(r, r_2) \right) \delta\alpha(r) ,$$  

(4)

where the differential operators $\Delta_k = 1 - (-1)^k \ell^* \hat{s}_k \cdot \nabla_{r_k}$ with $k = 1, 2$ and $\delta\alpha = c\delta\mu_\alpha$. Note that if the source and detector are oriented in the inward and outward normal directions, respectively, then (4) becomes

$$\phi(r_1, -\hat{n}(r_1); r_2, \hat{n}(r_2)) = \frac{c}{4\pi} \left( 1 + \frac{\ell^*}{\ell} \right)^2 \int d^3 r \left( G(r_1, r) G(r, r_2) - \frac{\ell^*}{3} \nabla_r G(r_1, r) \cdot \nabla_r G(r, r_2) \right) \delta\alpha(r) ,$$  

(5)

where we have used the boundary conditions on $G$ to evaluate the action of the $\Delta_k$ operators. Eq. (5) is the main result of this paper. It may be interpreted as providing corrections to the DA since the first term on the right hand side of (5) corresponds to the standard DA in an inhomogeneous absorbing medium. Analogous results for inhomogeneous scattering media will be presented elsewhere.

For the remainder of this paper we will work in the planar measurement geometry, often encountered in small-animal imaging. In this case, (4) becomes

$$\phi(\rho_1, \rho_2) = \int d^3 r K(\rho_1, \rho_2; r) \delta\alpha(r) ,$$  

(6)

where $\rho_1$ denotes the transverse coordinates of a point source in the plane $z = 0$, $\rho_2$ denotes the transverse coordinates of a point detector in the plane $z = L$, and the dependence of $\phi$ on $s_1$ and $s_2$ is not made explicit. Evidently, from considerations of invariance of $\phi$ under translations of the source and detector, the kernel $K(\rho_1, \rho_2; r)$ may be expressed as the Fourier integral

$$K(\rho_1, \rho_2; r) = \frac{1}{(2\pi)^2} \int d^2 q_1 d^2 q_2 K(q_1, q_2; z) \exp \left[ i(q_1 - q_2) \cdot \rho - i(q_1 \cdot \rho_1 - q_2 \cdot \rho_2) \right] ,$$
where \( r = (\rho, z) \). The function \( \kappa \) may be obtained from the plane-wave expansion of the diffusion Green's function obeying appropriate boundary conditions.

Inversion of the integral equation (6) may be carried out by analytic methods. These methods have been shown to be computationally efficient and may be applied to data sets consisting of a very large number of measurements. The approach taken is to construct the singular value decomposition of the linear operator \( K \) in the proper Hilbert space setting and then use this result to obtain the pseudoinverse solution to (6). In this manner, it is possible to account for the effects of sampling and limited data and thereby obtain the best (in the sense of minimizing the appropriate \( L^2 \) norm) bandlimited approximation to \( \delta \alpha \).

In conclusion, we have described a series of corrections to the usual formulation of the DA in optical tomography. We have found in simulations, which will be reported elsewhere, that these corrections give rise to superresolved images with resolution below \( \ell^* \). The effects of corrections are most significant in optically thin layers. However, corrections to the DA may also be expected to be important for thick layers when inhomogeneities in the absorption are located near the surface. Finally, we note that higher order corrections to the DA may be important for the nonlinear ISP and also in situations when simultaneous reconstruction of both \( \mu_a \) and \( \mu_s \) is of interest.

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REFERENCES