Tomography of Highly Scattering Media with the Method of Rotated Reference Frames

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MOTIVATION
(The Inverse Problem Perspective)
Given a data function $\phi(\rho_s, \rho_d)$ which is measured for multiple pairs $(\rho_s, \rho_d)$, find the absorption coefficient $\alpha(r)$ inside the slab.
Linearized Integral Equation

\[ \phi(\rho_s, \rho_d) = \int \Gamma(\rho_s, \rho_d; r) \delta \alpha(r) d^3r \]

\[ \phi(\rho_s, \rho_d) = \frac{I(\rho_s, z_s; \rho_d, z_d) - I_0(\rho_s, z_s; \rho_d, z_d)}{I_0(\rho_s, z_s; \rho_d, z_d)} \]

(measurable data-function)

\[ \Gamma(\rho_s, \rho_d) = G_0(\rho_s, z_s; r)G_0(r; \rho_d, z_d) \]

(first Born approximation)

\[ \alpha(r) = \alpha_0 + \delta \alpha(r) \]
How To Invert?

Numerical SVD approach:

\[ \phi_i = \sum_j \Gamma_{ij} \delta \alpha_j \]

\[ \phi = \Gamma \delta \alpha \]

\[ \Gamma^* \phi = \Gamma^* \Gamma \delta \alpha \]

\[ \delta \alpha^+ = \left( \Gamma^* \Gamma \right)_{reg}^{-1} \Gamma^* \phi \]
Analytical SVD approach: Making use of the translational invariance

\[ \tilde{\phi}(q_s, q_d) = \int \phi(\rho_s, \rho_d)e^{i(q_s \cdot \rho_s + q_d \cdot \rho_d)}d^2 \rho_s d^2 \rho_d \]

\[ q_s = q / 2 + p, \quad q_d = q / 2 - p; \]

Data function: \( \psi(q, p) = \tilde{\phi}(q / 2 + p, q / 2 - p) \)

\[ \psi(q, p) = \int_0^L g_s(q / 2 + p; z)g_d(q / 2 - p; z)\delta \tilde{\alpha}(q; z)dz \]

\[ \delta \tilde{\alpha}(q; z)dz = \int \delta \alpha(p, z)e^{ip \cdot q}d^2 \rho \]

\[ G_0(\rho_s, z_s; \rho, z) = \int \frac{d^2 q}{(2\pi)^2}g_s(q; z)e^{iq \cdot (\rho - \rho_s)} \]

\[ G_0(\rho, z; \rho_d, z_d) = \int \frac{d^2 q}{(2\pi)^2}g_d(q; z)e^{iq \cdot (\rho_d - \rho)} \]
Imaging complex structures with diffuse light
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In the case of RTE, we need the plane-wave decomposition of the Greens function, of the form

\[
G_0(r, \hat{s}; r', \hat{s'}) = \int \frac{d^2 q}{(2\pi)^2} g(q; z, \hat{s}; z', \hat{s'}) e^{iq(\rho - \rho')}
\]

and the integral kernel

\[
\Gamma(q, p; z) = \int g(q / 2 + p; z_s, \hat{z}; z, \hat{s}) g(q / 2 - p; z, \hat{s}; z_d, \hat{z}) d^2 s
\]

We then get the integral equations of the form

\[
\psi(q, p) = \int_{z_s}^{z_d} \Gamma(q, p; z) \delta \tilde{\alpha}(q; z) dz
\]
Motivation Continued
(The Forward Model Perspective)
<table>
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<th>Plane Wave Modes</th>
<th>The Weyl Expansion</th>
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</table>
| **The Helmholtz Equation** \((\nabla^2 + k_0^2)u = 0\)  
\(k = \text{const}\)  
\(e^{ik \cdot r}\)  
\(k \cdot k = k_0^2\)  
\(\frac{e^{ik_0r}}{r} = \frac{i}{2\pi} \int Q^{-1} e^{i(q \cdot p + Qz)} d^2q\)  
\(Q = \sqrt{k_0^2 - q^2}\)  |  |
| **The Diffusion Equation**  
\((-D\nabla^2 + \alpha)u = 0\)  
\(D, \alpha = \text{const}\)  
\(e^{-k \cdot r}\)  
\(k \cdot k = k_0^2 = \alpha / D\)  
\(\frac{e^{-k_0r}}{r} = \frac{1}{2\pi} \int Q^{-1} e^{i(q \cdot p - Qz)} d^2q\)  
\(Q = \sqrt{k_0^2 + q^2}\)  |  |
| **RTE**  
\((\hat{s} \cdot \nabla + \mu_t)I(r, \hat{s}) = \mu_s \int A(\hat{s}, \hat{s}')I(r, \hat{s}')\)  
\(\mu_t, \mu_s = \text{const}\)  | Arnold Kim and us (the past \(~10\) years)  |
|  | Arnold Kim and us (the past \(~10\) years)  |
Spectral Method for the RTE?

RTE: \((\hat{s} \cdot \nabla + \mu_t)I(\mathbf{r}, \hat{s}) = \mu_s \int A(\hat{s}, \hat{s}')I(\mathbf{r}, \hat{s}')d^2\hat{s}' + \varepsilon(\mathbf{r}, \hat{s})\)

Where is the "spectral variable"?

How can we write this equation in the form \((z + W)\left| I \rightangle = \left| \varepsilon \rightangle\) ?

\(\mu_t\) and \(\mu_s\) do not qualify...

We can try to expand \(I(\mathbf{r}, \hat{s})\) into a 3D Fourier integral with respect to \(\mathbf{r}\) and into the basis of ordinary spherical harmonics \(Y_{lm}(\theta, \varphi)\) with respect to \(\hat{s}\) ...

...and see if the equation can be cast into the desired form.
The Conventional Method of Spherical Harmonics

This results in the following system of equations with respect to the vector of the expansion coefficients $|I(k)\rangle$ ($k$ - the Fourier variable):

$$iA^{(x)}k_x |I(k)\rangle + iA^{(y)}k_y |I(k)\rangle + iA^{(z)}k_z |I(k)\rangle + D|I(k)\rangle = |\varepsilon(k)\rangle$$

$A^{(x)}$, $A^{(y)}$, $A^{(z)}$ are different matrices.

$$A^{(x)}_{lm,l'm'} = \int \sin\theta \cos\phi Y^*_l m(\theta,\phi) Y^{l'}m'(\theta,\phi) \sin\theta d\theta d\phi \quad \text{etc. .....}$$

“This rather awe-inspiring set of equations … has perhaps only academic interest”.

K.M. Case, P.F. Zweifel, Linear Transport Theory
Rotated Reference Frames

The usual spherical harmonics are defined in the laboratory reference frame. Then $\theta$ and $\varphi$ are the polar angles of the unit vector $\hat{s}$ in that frame.

**THE MAIN IDEA:** For each value of the Fourier variable $k$, use spherical harmonics defined in a reference frame whose z-axis is aligned with the direction of $k$. We call such frames "rotated". Spherical harmonics defined in the rotated frame are denoted by $Y(\hat{s}; \hat{k})$. 
Rotation of the Laboratory Frame $(x,y,z)$.

$$Y_{lm}(\hat{s};\hat{k}) = \sum_{m'=-l}^{l} D_{m'm}^{l}(\phi_{k},\theta_{k},0)Y_{lm'}(\hat{s})$$

Wigner D-functions

Euler angles

Spherical functions in the laboratory frame
$(\mathbf{s} \cdot \nabla + \mu_t)I = \mu_s A I + \varepsilon$

$I(\mathbf{r}, \mathbf{s}) = \int \tilde{I}(\mathbf{k}, \mathbf{s}) e^{i\mathbf{k} \cdot \mathbf{r}} d^3 k$

$\varepsilon(\mathbf{r}, \mathbf{s}) = \int \tilde{\varepsilon}(\mathbf{k}, \mathbf{s}) e^{i\mathbf{k} \cdot \mathbf{r}} d^3 k$

$(i\mathbf{k} \cdot \mathbf{s} + \mu_t)\tilde{I} = \mu_s A \tilde{I} + \tilde{\varepsilon}$

$\tilde{I}(\mathbf{k}, \mathbf{s}) = \sum_{l,m} F_{lm}(\mathbf{k}) Y_{lm}(\mathbf{s}; \mathbf{k})$

$\tilde{\varepsilon}(\mathbf{k}, \mathbf{s}) = \sum_{l,m} E_{lm}(\mathbf{k}) Y_{lm}(\mathbf{s}; \mathbf{k})$

$A(\mathbf{s}, \mathbf{s'}) = \sum_{l,m} A_l Y_{lm}(\mathbf{s}; \mathbf{k}) Y_{lm}^*(\mathbf{s}; \mathbf{k})$
\[ i k \sum_{l'm'} R_{lm}^{l'm'} F_{l'm'}(k) + \sigma_{l} F_{lm}(k) = E_{lm}(k) \]

\[ \sigma_{l} = \mu_{a} + \mu_{s} (1 - A_{l}) \]

\[ R_{lm}^{l'm'} = \int \hat{s} \cdot \hat{k} \ Y_{lm}^* (\hat{s}; \hat{k}) Y_{l'm'} (\hat{s}; \hat{k}) d^{2}s = \]

\[ = \delta_{mm'} \left[ b_{l}(m) \delta_{l'=l-1} + b_{l+1}(m) \delta_{l'=l+1} \right] \]

\[ b_{l}(m) = \sqrt{\frac{l^2 - m^2}{4l^2 - 1}} \]
RTE in the Angular Basis of Rotated Spherical Functions

\[ i k R |I(k)\rangle + D |I(k)\rangle = |E(k)\rangle \]

- **Scalar spectral parameter**
- **Block-tridiagonal real symmetric matrix**
- **Diagonal matrix**
  \[ S_{lm,l'm'} = \delta_{ll'} \delta_{mm'} [\mu_a + \mu_s (1 - A_l)] \]
- **Source term**
- **Parameters of the phase function. (For the HG model, \( A_l = g^l \))**
Let $D = SS$

$$W = S^{-1}RS^{-1}$$

$$ikR \left| I(k) \right\rangle + D \left| I(k) \right\rangle = \left| E(k) \right\rangle$$

$$(ikW + 1)S \left| I(k) \right\rangle = S^{-1} \left| E(k) \right\rangle$$

$$\left| I(k) \right\rangle = S^{-1} (1 + ikW)^{-1} S^{-1} \left| E(k) \right\rangle$$

$$W \left| \psi_{\mu} \right\rangle = \lambda_{\mu} \left| \psi_{\mu} \right\rangle$$

$$\left| I(k) \right\rangle = \sum_{\mu} S^{-1} \left| \psi_{\mu} \right\rangle \left\langle \psi_{\mu} \right| S^{-1} \left| E(k) \right\rangle \frac{1}{1 + ik \lambda_{\mu}}$$
The Spectral Solution

\[ I(\mathbf{r}, \hat{\mathbf{s}}) = \sum_{lm} \frac{1}{\sqrt{\sigma_l}} \int \sum_{\mu} \frac{\langle lm | \psi_\mu \rangle \langle \psi_\mu | S^{-1} | E(\mathbf{k}) \rangle Y_{lm}(\hat{\mathbf{s}}; \mathbf{k})}{1 + ik\lambda_\mu} e^{ik \cdot \mathbf{r}} d^3k \]

\[ \varepsilon(\mathbf{r}, \hat{\mathbf{s}}) = \delta(\mathbf{r} - \mathbf{r}_0) \delta(\hat{\mathbf{s}} - \hat{\mathbf{s}}_0) \]

\[ E_{lm}(\mathbf{k}) = \frac{1}{(2\pi)^3} e^{-ik \cdot \mathbf{r}_0} Y_{lm}^*(\hat{\mathbf{s}}_0 ; \mathbf{k}) \]

The integral is not easy… but doable.

\[
G_0(\mathbf{r}, \mathbf{s}; \mathbf{r}_0, \mathbf{s}_0) = \sum_{m=-\infty}^{\infty} \sum_{l,l'=|m|}^{\infty} Y_{lm}(\mathbf{s}; \hat{\mathbf{R}}) \chi_{ll'}^m(R) Y_{lm}^*(\mathbf{s}_0; \hat{\mathbf{R}})
\]

\[
\chi_{ll'}^m(R) = \frac{(-1)^m}{2\pi \sqrt{\sigma_l \sigma_{l'}}} \sum_{M=-\bar{l}}^{\bar{l}} (-1)^M \sum_{n, \lambda_n > 0} \frac{\langle l | \phi_n(M) \rangle \langle \phi_n(M) | l' \rangle}{\lambda_n^3(M)} \lambda_n(M)
\]

\[
\bar{l} = \min(l, l')
\]

\[
\mu = (M, n)
\]

\[
\langle lm | \psi_{Mn} \rangle = \delta_{mM} \langle l | \phi_n(M) \rangle
\]
The Plane-Wave Decomposition

\[
I(\mathbf{r}, \mathbf{s}) = \sum_{lm} \frac{1}{\sqrt{\sigma_l}} \left\langle l m | \psi_\mu \right| \left| \psi_\mu \right\rangle S^{-1} | E(k) \rangle Y_{lm}(\mathbf{s}'; \mathbf{k}) \frac{e^{i k \cdot r}}{1 + i k \lambda_\mu} d^3 k
\]

\[
G_0(\mathbf{r}, \mathbf{s}; \mathbf{r}_0, \mathbf{s}_0) = \sum_{lm, l'm'} \int \frac{d^2 q}{(2\pi)^2} e^{i q \cdot (\mathbf{r} - \mathbf{r}_0)} Y_{lm}(\mathbf{s}; \mathbf{z}) g_{lm'}(\mathbf{q}; \mathbf{z}, \mathbf{z}_0) Y_{l'm'}^*(\mathbf{s}_0; \mathbf{z})
\]

\[
g(\mathbf{q}; \mathbf{z}, \mathbf{z}_0) = \int_{-\infty}^{\infty} \frac{d k_z}{2\pi} e^{i k_z (z - z_0)} \mathcal{D}(\mathbf{q} + \hat{z} k_z) K(\sqrt{q^2 + k_z^2}) \mathcal{D}^\dagger(\mathbf{q} + \hat{z} k_z)
\]

\[
K(w) = \sum_{\mu} \frac{S^{-1} | \psi_\mu \rangle \left\langle \psi_\mu \right| S^{-1}}{1 + i w \lambda_\mu} \quad ; \quad \mathcal{D}(\mathbf{k}) = e^{-i \varphi_k J_z} e^{-i \theta_k J_y}
\]

\[
\cos \theta_k = k_z / \sqrt{q^2 + k_z^2}
\]
\begin{align*}
g_{l'm'}(q; z, z_0) &= \frac{e^{-i(m-m')\phi_q}}{\sqrt{\sigma_l \sigma_{l'}}} \left[ \text{sgn}(z - z_0) \right]^{l+l'+m+m'} \sum_{m_1=-l}^{l} \sum_{m_2=-l'}^{l'} \sum_{\mu, \lambda_\mu > 0} d^l_{m_1} [i\tau(q\lambda_\mu)] \\
&\quad \times \langle lm_1 | \psi_\mu \rangle \frac{e^{-Q_\mu(q)|z-z_0|}}{\lambda_\mu^2 Q_\mu(q)} \langle \psi_\mu | l'm_2 \rangle d^{l''}_{m_1 m_2} [i\tau(q\lambda_\mu)]
\end{align*}

\[Q_\mu(q) = \sqrt{q^2 + 1 / \lambda_\mu^2}\]

\[\cos[i\tau(x)] = \sqrt{1 + x^2}, \quad \sin[i\tau(x)] = -ix\]

\[D^l_{m'm'}(\alpha, \beta, \gamma) = e^{-im\alpha} d^l_{mm'}(\beta)e^{-im'\gamma}\]
Evanescent Waves

Plane waves:

\[ I_k = e^{-k \cdot r} F_k(\mathbf{\hat{s}}), \quad \mathbf{k} \cdot \mathbf{k} = 1 / \lambda_{\mu}^2, \quad \mathbf{k} = k \mathbf{\hat{n}} \]

\( \mathbf{\hat{n}} \) - real unit vector

\[ F_k(\mathbf{\hat{s}}) = \sum_{lm} \langle lm \mid \psi_{\mu} \rangle Y_{lm}(\mathbf{\hat{s}}; \mathbf{\hat{k}}) \]

Evanescent waves:

\[ \mathbf{k} = -i\mathbf{q} \pm \hat{\mathbf{z}} \sqrt{q^2 + 1 / \lambda_{\mu}^2}, \quad \mathbf{q} \cdot \hat{\mathbf{z}} = 0, \quad \mathbf{k} \cdot \mathbf{k} = 1 / \lambda_{\mu}^2 \]
\[ I_{kMn} = \exp \left( -\frac{\mathbf{k} \cdot \mathbf{r}}{\lambda_{Mn}} \right) \sum_{lm} Y_{lm}(\mathbf{s}; \mathbf{z}) \frac{\exp(-im\varphi_k)}{\sqrt{\sigma_l}} d^l_{mM}(\theta_k) \langle l | \phi_n(M) \rangle \]  

(plane wave modes)

\[ I_{qMn}^{(\pm)} = \exp \left[ i\mathbf{q} \cdot \mathbf{p} - Q_{Mn}(q)z \right] \sum_{lm} Y_{lm}(\mathbf{s}; \mathbf{z}) \frac{\exp(-im\varphi_q)}{\sqrt{\sigma_l}} (-1)^{l+m} \frac{1}{{\mathcal{J}}_{m-M}(\theta_k)} \langle l | \phi_n(M) \rangle \]  

(evanescent modes)

\[ G_0(r, \mathbf{s}; r_0, \mathbf{s}_0) = \sum_{Mn} \int \frac{d^2q}{(2\pi)^2} I_{qMn}^{(\pm)} (r, \mathbf{s}) V_{qMn} I_{-qMn}^{(\mp)} (r_0, -\mathbf{s}_0) \]

\[ V_{qMn} = \frac{1}{Q_{Mn}(q)\lambda_{Mn}^2} \]
Half-space problem

$z > 0$
Scattering medium
Propagation described by the RTE

$z < 0$
Non-scattering medium

$$I = \int \frac{d^2 q}{(2\pi)^2} \sum_{Mn} A_{Mn}(q) I_{qMn}^{(+)}$$
\[ \mu_a / \mu_s = 6.0 \times 10^{-5}, \quad g = 0.98. \]
RESULTS: SIMULATIONS
A set of 5 point absorbers in an $L=6\ell^*$ slab
The field of view is $16\ell^*$

RTE

Diffusion approximation
A bar target in the center of the same slab

RTE Diffusion approximation
RESULTS: EXPERIMENT
Two thin vertical wires in a 1cm thick slab filled with intralipid solution (looks like milk)

RTE Diffusion approximation
CONCLUSIONS

- The method of rotated reference frames can be used in optical tomography of mesoscopic samples.
- The images are of superior quality compared to those obtained by using the diffusion approximation.
- The quality of images can be comparable to that in X-ray tomography because the RTE retains some information about ballistic rays, single-scattered rays, etc.


