Tomography of Highly Scattering Media with the Method of Rotated Reference Frames

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Outline
Motivation
(The Forward Problem Perspective)
<table>
<thead>
<tr>
<th>Plane Wave Modes</th>
<th>The Weyl Expansion</th>
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</table>
| **The Helmholtz Equation**  
\((\nabla^2 + k_0^2)u = 0\)  
\(k = \text{const}\)  
\(e^{ik \cdot r}\)  
\(k \cdot k = k_0^2\)  
\(\frac{e^{ik_0r}}{r} = \frac{i}{2\pi} \int Q^{-1} e^{i(q \cdot p + Qz)} d^2q\)  
\(Q = \sqrt{k_0^2 - q^2}\)  
| **The Diffusion Equation**  
\((-\nabla \cdot D \nabla + \alpha)u = 0\)  
\(D, \alpha = \text{const}\)  
\(e^{-k \cdot r}\)  
\(k \cdot k = k_0^2 = \alpha / D\)  
\(\frac{e^{-k_0r}}{r} = \frac{1}{2\pi} \int Q^{-1} e^{i(p \cdot r - Qz)} d^2q\)  
\(Q = \sqrt{k_0^2 + q^2}\)  
| **RTE**  
\((\hat{s} \cdot \nabla + \mu_t)I(r, \hat{s}) = \mu_s \int A(\hat{s}, \hat{s}')I(r, \hat{s}')\)  
\(\mu_t, \mu_s = \text{const}\)  
| Not known  
Not known  
Not known |
MOTIVATION
(The Inverse Problems Perspective)
Given a data function $\phi(\rho_s, \rho_d)$ which is measured for multiple pairs $(\rho_s, \rho_d)$, find the absorption coefficient $\alpha(r)$ inside the slab.
Linearized Integral Equation

\[ \phi(p_s, p_d) = \int \Gamma(p_s, p_d; r) \delta \alpha(r) d^3r \]

\[ \phi(p_s, p_d) = \frac{I(p_s, z_s; p_d, z_d) - I_0(p_s, z_s; p_d, z_d)}{I_0(p_s, z_s; p_d, z_d)} \]

(measurable data-function)

\[ \Gamma(p_s, p_d) = G_0(p_s, z_s; r)G_0(r; p_d, z_d) \]

(first Born approximation)

\[ \alpha(r) = \alpha_0 + \delta \alpha(r) \]
Analytical SVD approach: Making use of the translational invariance

\[ \tilde{\phi}(q_s, q_d) = \int \phi(p_s, p_d)e^{i(q_s \cdot p_s + q_d \cdot p_d)} d^2\rho_s d^2\rho_d \]

\[ q_s = q / 2 + p, \quad q_d = q / 2 - p; \]

Data function: \( \psi(q, p) = \tilde{\phi}(q / 2 + p, q / 2 - p) \)

\[ \psi(q, p) = \int_0^L g_s(q / 2 + p; z)g_d(q / 2 - p; z) \delta \tilde{\alpha}(q; z) dz \]

\[ \delta \tilde{\alpha}(q; z) dz = \int \delta \alpha(p, z)e^{i\rho \cdot q} d^2\rho \]

\[ G_0(p_s, z_s; p, z) = \int \frac{d^2q}{(2\pi)^2}g_s(q; z)e^{iq \cdot (p - p_s)} \]

\[ G_0(p, z; p_d, z_d) = \int \frac{d^2q}{(2\pi)^2}g_d(q; z)e^{iq \cdot (p_d - p)} \]
Imaging complex structures with diffuse light
Optics Express 16(7), 5048-5060 (2008)
In the case of RTE, we need the plane-wave decomposition of the Greens function, of the form

\[ G_0(\mathbf{r}, \mathbf{s}; \mathbf{r}', \mathbf{s}') = \int \frac{d^2q}{(2\pi)^2} g(q; z, \mathbf{s}; z', \mathbf{s}') e^{iq(\mathbf{r}-\mathbf{r}')} \]

and the integral kernel

\[ \Gamma(\mathbf{q}, \mathbf{p}; z) = \int g(q / 2 + \mathbf{p}; z_s, \mathbf{z}; z, \mathbf{s}) g(q / 2 - \mathbf{p}; z, \mathbf{s}; z_d, \mathbf{z}) d^2s \]

We then get the integral equations of the form

\[ \psi(q, p) = \int_{z_s}^{z_d} \Gamma(q, p; z) \delta \tilde{\alpha}(q; z) dz \]
THE METHOD
Rotated Reference Frames

The usual spherical harmonics are defined in the laboratory reference frame. Then $\theta$ and $\phi$ are the polar angles of the unit vector $\hat{s}$ in that frame.

**THE MAIN IDEA:**

For each value of the Fourier variable $k$, use spherical harmonics defined in a reference frame whose z-axis is aligned with the direction of $k$.

We call such frames "rotated".

Spherical harmonics defined in the rotated frame are denoted by $Y(\hat{s};\hat{k})$. 
Rotation of the Laboratory Frame 
\((x,y,z)\).

\[ Y_{lm}(\hat{s};\hat{k}) = \sum_{m'=-l}^{l} D_{m'm}^{l}(\varphi_{k}, \theta_{k}, 0) Y_{lm'}(\hat{s}) \]

Wigner D-functions

Euler angles

Spherical functions in the laboratory frame
Advantages of the Method

- Numerical problem is reduced to diagonalization of a set of tridiagonal matrices
- Once the eigenvalues and eigenvectors are computed, the Green’s function, the plane-wave modes and the Weyl expansion for the GF can be obtained in analytical form
- The Weyl expansion can also be obtained in a slab with appropriate boundary conditions
RESULTS: SIMULATIONS
A set of 5 point absorbers in an L=6ℓ* slab
The field of view is 16ℓ*

RTE

Diffusion approximation
A bar target in the center of the same slab

RTE

Diffusion approximation
RESULTS: SIMULATIONS
Two thin vertical wires in a 1cm thick slab filled with intralipid solution (looks like milk)

RTE

Diffusion approximation


Available on the web at  
http://whale.seas.upenn.edu/vmarkel/papers.html
CONCLUSIONS

- The method of rotated reference frames can be used in optical tomography of mesoscopic samples.
- The images are of superior quality compared to those obtained by using the diffusion approximation.
- The quality of images can be comparable to that in X-ray tomography because the RTE retains some information about ballistic rays, single-scattered rays, etc.