BROKEN-RAY TOMOGRAPHY

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Zero Scattering Regime:
Conventional X-Ray Tomography
Strong Scattering Regime: Diffuse Optical Tomography

- Many source-detector pairs
- Severely ill-posed IP
- Nonlinear IP
Mesoscopic Scattering Regime: Single-Scattering Tomography
## SSOT And Other Modalities

<table>
<thead>
<tr>
<th></th>
<th>Linear IP</th>
<th>IP is MILDLY Ill-posed</th>
<th>Single projection</th>
<th>Reflection geometry</th>
<th>Nonionizing radiation</th>
<th>Quantitative images</th>
<th>Reconstruction of scattering and absorption</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>X-ray CT</strong></td>
<td>Y (if E=const)</td>
<td>Y</td>
<td>N</td>
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<td>Y</td>
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<tr>
<td><strong>DOT</strong></td>
<td>N</td>
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<td>Y</td>
<td>Y</td>
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<td>?</td>
<td>Y (with time or freq.)</td>
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<tr>
<td><strong>SSOT</strong></td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y (with two det. angles)</td>
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</table>
The Broken-Ray Integral Transform

(a) \( \mu_s = \text{const} \) (and is known)
\[
\int_{\text{SSR}} \mu_t[r(\ell)]d\ell = \phi(r_1, \hat{s}_1; r_2, \hat{s}_2)
\]

(b) \( \mu_s \neq \text{const} \) (and is unknown)
\[
\int_{\text{SSR}} \mu_t[r(\ell)]d\ell - \ln \left[ \frac{\mu_s(R_{12})}{\langle \mu_s \rangle} \right] = \phi(r_1, \hat{s}_1; r_2, \hat{s}_2)
\]

The measurable data function:
\[
\phi(r_1, \hat{s}_1; r_2, \hat{s}_2) = -\ln \left[ \frac{r_{12} \sin \theta_1 \sin \theta_2 I_{\text{measured}}}{\langle \mu_s \rangle A(\hat{s}_1, \hat{s}_2) I_{\text{incident}}} \right]
\]
Broken rays: You can go two ways

1) A → C
   B → D

2) A → D
   B → C
Outline:

1) Numerical test (40x40 rays per slice), full RTE forward solver, no inverse crime

2) Generalized filtered back-projection formula

3) Inverse crime simulations based on this formula (many rays)
PART 1: Numerical simulations: Data from the RTE (no inverse crime)

- Forward model based on the RTE
- Isotropic scattering
- FULL ACCOUNT OF MULTIPLE SCATTERING
- BOUNDARY CONDITIONS SATISFIED EXACTLY
- 3D integral equation for density discretized on a rectangular grid
- Direct inversion of a well-posed square matrix
- Mathematical details on next page...
\[
\mu_t(r) \cdot \nabla + \mu_a(r) + \mu_s(r)] I(r, \hat{s}) = \mu_s(r) \int \frac{1}{4\pi} I(r, \hat{s}') d^2\hat{s}'
\]

\[
u(r) = \int I(r, \hat{s}) d^2\hat{s}
\]

\[
u(r) = \nu_b(r) + \int g_b(r, r') \frac{\mu_s(r')}{4\pi} u(r') d^3r'
\]

\[
I(r, \hat{s}) = I_b(r, \hat{s}) + \int G_b(r, \hat{s}; r', \hat{s}') \frac{\mu_s(r')}{4\pi} u(r') d^3r' d^2\hat{s}'
\]

\[
G_b(r, \hat{s}; r', \hat{s}') = g_b(r, r') \delta \left( \hat{u}(r - r') - \hat{s}' \right) \delta (\hat{s} - \hat{s}')
\]

\[
g_b(r, r') = \int G_b(r, \hat{s}; r', \hat{s}') d^2\hat{s} d^2\hat{s}' = \exp \left[ \frac{1}{\|r - r'\|^2} \int_0^{\|r - r'\|} \mu_t \left( r' + \ell \hat{u}(r - r') \right) d\ell \right]
\]
34 source beams per slice

34 detectors per source

$L_y = 122h$

$L_x = 11h$ or $25h$

$L_z = 40h$

Total attenuation $\mu_t$ is reconstructed in each slice.
ABSORPTION

Background:
\[ \mu_a h = 0.01 \]
Targets:
\[ 0.06 < \mu_a h < 0.2 \]

\[ \mu_s h = 0.04 \]
\[ \mu_s L_z = 1.6 \]

\[ \mu_s h = 0.08 \]
\[ \mu_s L_z = 3.2 \]
Reconstruction of scattering+absorption in a thicker sample: $L_x=25h$

34 source beams per slice

$L_y = 122h$

$L_x = 25h$

$L_z = 40h$

34 detectors per source at this angle
Simultaneous reconstruction of absorption and scattering

**SCATTERING**
- Background: $\bar{\mu}_s L_z = 1.6$
- Targets: $1.33 \mu_s \leq \mu_s \leq 2 \mu_s$

**ABSORPTION**
- Background: $\bar{\mu}_a = 0.1 \bar{\mu}_s$
- Targets: $2 \mu_a \leq \mu_a \leq 5 \mu_a$
Inhomogeneities with stronger scattering

### SCATTERING

- **Background:** $\bar{\mu}_s L_z = 1.6$
- **Targets:** $2\mu_s \leq \mu_s \leq 3\mu_s$

### ABSORPTION

- **Background:** $\bar{\mu}_a = 0.1\bar{\mu}_s$
- **Targets:** $2\mu_a \leq \mu_a \leq 5\mu_a$

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### SCATTERING

- Background: $\mu_s L_z = 1.6$
- Targets: $2\mu_s \leq \mu_s \leq 3\mu_s$

### ABSORPTION

- Background: $\mu_a = \mu_s$
- Targets: $2\mu_a \leq \mu_a \leq 5\mu_a$

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Same as before, but larger optical depth

**SCATTERING**
- Background: $\bar{\mu}_s L_z = 3.2$
- Targets: $2\mu_s \leq \mu_s \leq 3\mu_s$

**ABSORPTION**
- Background: $\bar{\mu}_a = \bar{\mu}_s$
- Targets: $2\mu_a \leq \mu_a \leq 5\mu_a$
PART 2: FBF derivation

\[ \int_{SSR(w,\Delta)} \mu_i[r(\ell)] d\ell = \phi(w,\Delta) \]

\[ r = (y, z) \]

\[ y(\ell) = w + \eta(\Delta, \ell) \]

\[ z(\ell) = \zeta(\Delta, \ell) \]
\[ \eta(\Delta, \ell) = \begin{cases} 
0, & \ell < L_1(\Delta) \\
\ell - L_1(\Delta), & L_1(\Delta) < \ell < L_1(\Delta) + L_2(\Delta) 
\end{cases} \]

\[ \zeta(\Delta, \ell) = \begin{cases} 
\ell, & \ell < L_1(\Delta) \\
L_1(\Delta) + [\ell - L_1(\Delta)]\cos(\beta), & L_1(\Delta) < \ell < L_1(\Delta) + L_2(\Delta) 
\end{cases} \]

\[ L_1(\Delta) = L - \Delta \ctg(\beta) , \quad L_2(\Delta) = \Delta / \sin(\beta) \]

The ray length (within the medium): \( L_1(\Delta) + L_2(\Delta) \)
Fourier slice theorem

\[ \tilde{\phi}(k, \Delta) = \int_{0}^{L_1(\Delta)} e^{ik\eta(\Delta, \ell)} \tilde{\mu}_t[k, \zeta(\Delta, \ell)]d\ell \]

\[ \tilde{\phi}(k, \Delta) = \int_{0}^{L_1(\Delta)} \tilde{\mu}_t(k, \ell)d\ell + \frac{e^{ikL_1(\Delta)\tan\beta}}{\cos\beta} \int_{L_1(\Delta)}^{L} \tilde{\mu}_t(k, \ell)e^{-ik\ell\tan\beta}d\ell \]

Define new variables:  \( q = k \tan\beta \);  \( c = \cos\beta \)

\[ f(z) = \tilde{\mu}_t(q \cot\beta, z) \]

\[ F(z) = \tilde{\phi}[q \cot\beta,(L - z) \tan\beta] \]
The inverse solution:

\[
\int_0^z f(\ell) d\ell + \frac{1}{c} e^{iqz} \int_z^L e^{-iq\ell} f(\ell) d\ell = F(z)
\]

\[c = \cos \beta, \quad 0 < z < L\]

\[
f_i(z) = -\kappa \left[ G(z) - i\kappa qe^{-i\kappa qz} \int_0^z e^{i\kappa \ell} G(\ell) d\ell \right]
\]

\[\kappa = \frac{c}{1-c} \quad ; \quad G(z) = \left( \frac{d}{dz} - iq \right) F(z)\]
Putting everything together...

\[
\tilde{\mu}_t(k, z) = \sigma \left\{ H(k, z) - ik\sigma e^{-i k \sigma z} \int_{0}^{z} e^{i \sigma k \ell} H(k, \ell) d\ell \right\}
\]

\[
\sigma = \text{ctg} \left( \frac{\beta}{2} \right); \quad \text{ctg} = \frac{\cos \beta}{\sin \beta}
\]

\[
H(k, z) = \left( \frac{\partial}{\partial \Delta} + ik \right) \tilde{\phi}(k, \Delta) \bigg|_{\Delta = (L - z) \text{tg} \beta}
\]
The real-space inversion formula

\[ \mu_t(y, z) = \int_{-\infty}^{\infty} \tilde{\mu}_t(k, z) e^{-iky} \frac{dk}{2\pi} \]

\[ \mu_t(y, z) = \sigma \left\{ \left( \frac{\partial}{\partial \Delta} - \frac{\partial}{\partial y} \right) \phi(y, \Delta) + \tau \frac{\partial}{\partial y} \left[ \phi(y + \sigma z, L \tg \beta) - \phi(y, \Delta) \right] \right\} \]

\[ - (1 + \tau) \frac{\partial}{\partial y} \int_{\Delta}^{L \tg \beta} \phi(y + \tau(\ell - \Delta), \ell) d\ell \]

\[ \sigma = \ctg \left( \frac{\beta}{2} \right) ; \quad \tau = \ctg \left( \frac{\beta}{2} \right) / \tg(\beta) \]
PART 3: Inverse-crime simulations

Reconstruction using the Fourier-space formula

\[ \beta = \frac{\pi}{4} \]

Model \hspace{1cm} L/h=40 \hspace{1cm} L/h=400
Reconstruction of a Gaussian

\[ \frac{w}{L} = \frac{3}{40} \quad \frac{w}{L} = \frac{5}{40} \quad \frac{w}{L} = \frac{7}{40} \quad \frac{w}{L} = \frac{10}{40} \]

\[ \beta = \frac{\pi}{4} \]

\[ h = 0.3 \]

\[ \frac{L}{40} \]
Profiles of the reconstruction of a Gaussian
Real-space formula (for Gaussians)
Simultaneous reconstruction of absorption and scattering

\[ \phi = \phi_1 - \phi_2 \]

(b) \( \mu_s \neq \text{const} \) (and is unknown)

\[
\int_{\text{SSR}} \mu_s[r(\ell)]d\ell - \ln \left[ \frac{\mu_s(R_{12})}{\langle \mu_s \rangle} \right] = \phi(r_1, s_1; r_2, s_2)
\]
Fourier-space image reconstruction formula for two rays

\[ \tilde{\mu}_t(k, z) = \frac{\sin \beta}{2} \left( ik - \frac{1}{ik} \frac{\partial^2}{\partial \Delta^2} \right) \tilde{\phi}(k, \Delta) \bigg|_{\Delta = (L-z) \tan \beta} \]
<table>
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<th>( \mu_t )</th>
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Model, \( w=7L/40 \)

Rec.

Model, \( w=3L/40 \)

Rec.

Model, \( w=1L/40 \)

Rec.

\[
\frac{\mu_s}{\mu_a} = \frac{4}{3}; \quad \delta \mu_{a,s} = \mu_{a,s} \exp \left[ -\frac{(r - r_{a,s})^2}{w^2} \right]; \quad \frac{h}{L} = \frac{0.3}{40}
\]
Same as above but for $\frac{\mu_s}{\mu_a} = 1$
SUMMARY

• SSOT allows accurate quantitative reconstruction of the attenuation function.
• With additional measurements, scattering and absorption can be reconstructed separately.
• Ill-posedness of the inverse problem is very mild.
• Tomographic imaging is feasible up to about six scattering lengths, with the noise-to-signal level of about 3% or less.
Preliminary Experiment
Figure 6: Experimental measurements of the specific intensity as a function of the exit position on the slab surface for intralipid concentrations 0.02% (a) and 0.04% (b).